A note on the extended superconformal algebras associated with manifolds of exceptional holonomy

José M Figueroa-O'Farrill [a,1]

[a]⟨j.m.figueroa@qmw.ac.uk⟩
Department of Physics, Queen Mary and Westfield College, Mile End Road,
London E1 4NS, UK

Abstract

It was observed some time ago by Shatashvili and Vafa that superstring compactification on manifolds of exceptional holonomy gives rise to superconformal field theories with extended chiral algebras. In their paper, free field realisations are given of these extended superconformal algebras inspired by Joyce's constructions of such manifolds as desingularised toroidal orbifolds. The purpose of this note is to give another realisation of these algebras starting not from free fields, but from the superconformal algebras associated to Calabi–Yau manifolds. These superconformal algebras, originally studied by Odake, are extensions of the $N{=}2$ Virasoro algebra. For the case of G_2 holonomy, our realisation is inspired in the conjectured construction of such manifolds as a desingularisation of $(K \times S^1)/\mathbb{Z}_2$, where K is a Calabi–Yau 3-fold admitting an antiholomorphic involution. Similarly, for the case of $Spin(7)$ holonomy our realisation suggests a construction of such manifolds as desingularisations of $K'/\mathbb{Z}_2$, where K' is a Calabi-Yau 4-fold admitting an antiholomorphic involution.

1 Introduction

In the context of M-theory [2] and F-theory [13], compactification to four dimensions requires that we do so on manifolds of seven and eight dimensions, respectively. If we require supersymmetry in four-dimensions we are forced to compactify on manifolds admitting parallel spinors. This in turn constraints the holonomy group of the manifold to be contained in the isotropy group of the spinor. For M an irreducible riemannian n-manifold which is

not locally symmetric, the possible holonomy groups are those in Berger's list. Of the groups in that list, only $SU(n/2)$, $Sp(n/4)$, G_2 (for n=7) and $Spin(7)$ (for n=8) admit parallel spinors (see, for example, [15]). For M a 7-dimensional simply-connected manifold to admit the minimum (nonzero) number of parallel spinors, yielding the minimum number of four-dimensional supersymmetries, its holonomy group must be G_2. The same conditions on an 8-dimensional manifold singles out those with $Spin(7)$ holonomy. In both of these cases there is only one parallel spinor.

For traditional superstring compactification from ten to four dimensions, the desired holonomy is $SU(3)$. Thanks to Yau's solution of the Calabi conjecture, any Kähler n-fold with vanishing canonical class admits a unique metric of $SU(n)$ holonomy in the same Kähler class. This provides us with many examples of such manifolds. On the other hand, relatively few examples are known of compact manifolds of exceptional holonomy. It was not until two years ago that the first such manifolds were constructed by Joyce [6,7] by desingularising toroidal orbifolds.

Such manifolds are not just interesting in their own right, but their study is relevant for superstring phenomenology. Just as in the case of Calabi–Yau 3-folds before them, it is hoped that much can be learned by exploring the superconformal field theories they give rise to. A particularly fruitful approach is to study the orbifold limit, since although the geometry becomes singular, the conformal field theory does not. Moreover, it seems that the orbifold limit captures some of the information on the desingularisation process [10]; for example, different desingularisations seem to be correspond to the different conformal field theories associated with the same orbifold via the process of turning on discrete torsion [14]. This prompted the *generalised mirror conjecture* in [10], for which more evidence would be welcome.

The superconformal algebras arising from compactification on Calabi–Yau n-folds are well known [9]. They correspond to extensions of the N=2 Virasoro algebra by a complex field of dimension $n/2$. For n=1 it is an extension of the N=2 Virasoro algebra by a complex fermion and a complex boson, whereas for n=2 it becomes the (small) N=4 Virasoro algebra. This is expected since $SU(2) = Sp(1)$ and all 4-manifolds with $Sp(1)$ holonomy are hyperkähler. These algebras exist for generic values of the Virasoro central charge. For n=3 and n=4, the cases of interest in the present note, the Virasoro central charge is fixed to 9 and 12 respectively, corresponding to compactification manifolds of dimensions 6 and 8, respectively.

For compactifications on manifolds of exceptional holonomy, the resulting superconformal algebras are extensions of the N=1 Virasoro algebra by superfields of weights $\frac{3}{2}$ and 2 for G_2, and 2 for $Spin(7)$. These algebras were written down by Shatashvili and Vafa in [10] for the first time in the present context;

although classical versions of these algebras had appeared in [5]. The $Spin(7)$ algebra had been discovered previously in a different context [4].[2] The G_2 algebra exists only for $c = \frac{21}{2}$, corresponding to 7-dimensional compactification manifolds, whereas the $Spin(7)$ algebra belongs to a one-parameter family of superconformal algebras [4], of which the $c=12$ point is the interesting one in the present context.

The algebras in [10] were constructed in a free field realisation appropriate to the study of those manifolds which are presented as desingularised toroidal orbifolds; however other constructions may exist and it is desirable to understand them in the language of conformal field theory. The purpose of this note is to construct new realisations of this algebra which belie constructions of these manifolds starting from Calabi–Yau 3- and 4-folds.

This note is organised as follows. In Section 2 we describe the extended superconformal algebras of interest: the ones associated to manifolds of $SU(3)$, $SU(4)$, G_2 and $Spin(7)$ holonomies. In [6] a construction of compact manifolds with G_2 holonomy is conjectured, which consists in desingularising an orbifold $(K \times S^1)/\mathbb{Z}_2$, where K is a Calabi–Yau 3-fold admitting an antiholomorphic involution, and the generator σ of the $\mathbb{Z}_2$ acts as the involution on K, and as inversion on the circle. This suggests that there should be a realisation of the G_2 superconformal algebra in terms of the superconformal algebra associated to the Calabi–Yau 3-fold K, and to the circle. Roughly the geometric $\mathbb{Z}_2$ induces an automorphism of the superconformal algebra and inside the fixed subalgebra one finds a realisation of the algebra in [10]. This will shown in section 3. There we also show that if we take a Calabi–Yau 4-fold K' admitting an antiholomorphic involution σ, then the conformal field theory associated to the orbifold $K'/\langle\sigma\rangle$ embeds a superconformal subalgebra isomorphic to the $Spin(7)$ algebra. This suggests a construction of compact 8-manifolds with $Spin(7)$ holonomy obtained by desingularising the orbifold. After completion of this work, we became aware of a preprint [1] which mentions the large volume limit of this last embedding.

Throughout we use the notation $[A, B]_\ell$ to denote the residue of the ℓ-th order pole in the operator product expansion of the fields A and B:

$$A(z)B(w) = \sum_{\ell \ll \infty} \frac{[A, B]_\ell(w)}{(z - w)^\ell} \,,$$

and assume familiarity with the axiomatics of these brackets as explained, for example, in [11].

---

[2] The $Spin(7)$ algebra was also written down in [3], where the G_2 algebra appears for the first time. The G_2 algebra had also appeared in [8]. I thank Andreas Honecker for reminding me of this work.

2 The superconformal algebras

In this section we write down the superconformal algebras associated with compactifications on manifolds of holonomy $SU(3)$ and $SU(4)$ [9], and G_2 and $Spin(7)$ [10]. Let M be an irreducible manifold with holonomy group in the above list. Every parallel form on M gives rise to a generator of the algebra, by pulling the form back to the worldsheet using the fermions. Parallel forms are precisely the singlets under the holonomy group in the representation $\bigwedge^* T$ where T is the irreducible representation on tangent vectors. Hence group theory alone tells us all about the parallel forms (indeed tensors) on M; or alternatively we can use the parallel fermions to construct these forms as bispinors.

2.1 The superconformal algebra of a Calabi–Yau n-fold

For $SU(n)$ holonomy, the tangent vectors form an irreducible $2n$-dimensional representation T. Its complexification splits into $T_{\mathbb{C}} = T' \oplus T''$, where T' is the fundamental complex representation of $SU(n)$ and T'' is its complex conjugate. The parallel forms are in one-to-one correspondence with the $SU(n)$ singlets in $\bigwedge^* T$. These include the Kähler form $\omega \in T' \otimes T'' \subset \bigwedge^2 T$ and its powers, but also the real 2-dimensional representation $\bigwedge^n T' \oplus \bigwedge^n T''$, corresponding to the real and imaginary parts of a complex $(n,0)$-form Ω. In the large volume limit, we can write down the following generators relative to a complex coordinate basis:

$$
\begin{aligned}
J &= \tfrac{1}{2}\omega_{a\bar{b}}\psi^a\psi^{\bar{b}} \\
H &= \tfrac{1}{n!}\Omega_{a_1 a_2 \cdots a_n}\psi^{a_1}\psi^{a_2}\cdots\psi^{a_n} \\
\bar{H} &= \tfrac{1}{n!}\bar{\Omega}_{\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n}\psi^{\bar{a}_1}\psi^{\bar{a}_2}\cdots\psi^{\bar{a}_n} \ ,
\end{aligned}
\tag{1}
$$

which will define an extension of the $N{=}1$ Virasoro algebra written for the first time in [9].

Let us now write down these algebras. We let T and G denote the generators of the $N{=}1$ Virasoro algebra with central charge $3n$. We let J be a weight one superconformal primary normalised to $[\mathsf{J},\mathsf{J}]_2 = -n$. Together with its superpartner $\mathsf{G}' = [\mathsf{G},\mathsf{J}]_1$, they generate the $N{=}2$ Virasoro algebra. Now let A and B be $N{=}1$ superconformal primaries of weight $n/2$. They are to be understood as the generators corresponding to the real and imaginary parts of the field H in (1). They satisfy the following operator product expansion with J:

$$
[\mathsf{J},\mathsf{A}]_1 = -n\mathsf{B} \quad \text{and} \quad [\mathsf{J},\mathsf{B}]_1 = n\mathsf{A} \ .
$$

We let $C = [G, A]_1$ and $D = [G, B]_1$ denote their superpartners. Because Ω is actually antiholomorphic, $\partial\Omega = 0$, (A, C) and (B, D) are $N{=}2$ (anti)chiral superfields. This means that they obey the following operator product expansions with the second supersymmetry generator: $[G', A]_1 = -D$ and $[G', B]_1 = C$. The remaining operator product expansions are given in terms of the ones involving the primary fields A and B, so we give only these. The others can be reconstructed using the Jacobi-like identities of the $[-, -]_\ell$ brackets, or equivalently the associativity of the operator product expansion. It is here that we must distinguish between $n{=}3$ and $n{=}4$.

2.1.1   n=3

The following operator product expansions hold:

$$
\begin{array}{ll}
[A, A]_3 = -4 & [A, A]_1 = 2(JJ) \\
[B, B]_3 = -4 & [B, B]_1 = 2(JJ) \\
[A, B]_2 = -4J & [A, B]_1 = -2\partial J \\
[A, C]_2 = -2G & [A, C]_1 = -2(JG') \\
[A, D]_2 = -2G' & [A, D]_1 = -2(JG) \\
[B, C]_2 = -2G' & [B, C]_1 = 2(JG) \\
[B, D]_2 = 2G & [B, D]_1 = 2(JG') \ ,
\end{array}
$$

where the normal-ordered product (AB) is defined by $(AB) = [A, B]_0$. As it stands, the chiral algebra defined by these brackets is not associative. The Jacobi-like identities are only satisfied modulo the ideal generated by the weight $\frac{5}{2}$ fields

$$
N_{CY}^{(1)} = \partial A - (JB) \qquad \text{and} \qquad N_{CY}^{(2)} = \partial B + (JA) \ . \tag{2}
$$

These fields are null for this value of the central charge $c{=}9$.

2.1.2   n=4

Similarly in this case, the following operator product expansions hold:

$$
\begin{array}{lll}
[A, A]_4 = -8 & [A, A]_2 = -4(JJ) & [A, A]_1 = -4(\partial JJ) \\
[B, B]_4 = -8 & [B, B]_2 = -4(JJ) & [B, B]_1 = -4(\partial JJ) \\
[A, B]_3 = -8J & [A, B]_2 = 4\partial J & [A, B]_1 = -\frac{4}{3}(JJJ) + \frac{4}{3}\partial^2 J \\
[A, C]_3 = -4G & [A, C]_2 = -4(JG') & [A, C]_1 = 2(GJJ) - 2(\partial JG') \\
[A, D]_3 = 4G' & [A, D]_2 = -4(JG) & [A, D]_1 = -2(JJG') - 2(\partial JG) \\
[B, C]_3 = -4G' & [B, C]_2 = 4(JG) & [B, C]_1 = 2(JJG') + 2(\partial JG) \\
[B, D]_3 = -4G & [B, D]_2 = -4(JG') & [B, D]_1 = 2(GJJ) - 2(\partial JG') \ ,
\end{array}
$$

where the normal-ordered product associates to the left; that is, $(ABC) = (A(BC))$. Again the above operator product expansions are associative only modulo the ideal generated by the fields given in (2), which now have weight 3 and are null for $c=12$.

We should remark that it follows from the above brackets that the super-conformal algebras associated with a Calabi–Yau n-fold have an additional automorphism, corresponding to multiplying the complex fields $\mathsf{A} + i\mathsf{B}$ and $\mathsf{C} + i\mathsf{D}$ by the same phase.

2.2   The algebra associated with G_2 holonomy

In a 7-dimensional irreducible manifold of G_2 holonomy, the tangent vectors are in an irreducible representation T of G_2. Computing the singlets in $\bigwedge^* T$, we find that there is a unique parallel 3-form ϕ, a unique parallel 4-form $\star\phi$, and their product. In the large volume limit we can write the fields

$$\Phi = \tfrac{1}{3!}\phi_{ijk}\psi^i\psi^j\psi^k$$
$$\Phi^* = \tfrac{1}{4!}(\star\phi)_{ijkl}\psi^i\psi^j\psi^k\psi^l \ ,$$

which generate an extension of the $N{=}1$ Virasoro algebra.

Let us now write down this algebra. We let T and G be the generators of the $N{=}1$ Virasoro algebra with $c{=}\frac{21}{2}$. Let P be a superprimary field of weight $\frac{3}{2}$ with operator product expansion $[\mathsf{P},\mathsf{P}]_3 = -7$ and $[\mathsf{P},\mathsf{P}]_1 = 6\mathsf{X}$, which defines X. The field X has weight 2, but it is is not a primary since $[\mathsf{T},\mathsf{X}]_3 = -\frac{7}{4}$. In addition, we have

$$[\mathsf{P},\mathsf{X}]_2 = -\tfrac{15}{2}\mathsf{P} \quad [\mathsf{P},\mathsf{X}]_1 = -\tfrac{5}{2}\partial\mathsf{P} \ ,$$

and

$$[\mathsf{X},\mathsf{X}]_4 = \tfrac{35}{4} \quad [\mathsf{X},\mathsf{X}]_2 = -10\mathsf{X} \quad [\mathsf{X},\mathsf{X}]_1 = -5\partial\mathsf{X} \ .$$

It follows from these formulae that $\mathsf{G}' \equiv \pm\frac{i}{\sqrt{15}}\mathsf{P}$, and $\mathsf{T}' \equiv -\frac{1}{5}\mathsf{X}$ satisfy an $N{=}1$ Virasoro algebra with central charge $c'{=}\frac{7}{10}$ corresponding to the tricritical Ising model. We now define $\mathsf{K} = [\mathsf{G},\mathsf{P}]_1$ and $\mathsf{M} = [\mathsf{G},\mathsf{X}]_1$ as the superpartners of P and X respectively. Because P is a superconformal primary, K is a Virasoro primary of weight 2. On the other hand M is not a primary because neither is X. Instead we have $[\mathsf{T},\mathsf{M}]_3 = [\mathsf{G},\mathsf{X}]_2 = -\frac{1}{2}\mathsf{G}$ and in addition:

$$[\mathsf{G},\mathsf{M}]_4 = -\tfrac{7}{2} \qquad [\mathsf{G},\mathsf{M}]_2 = \mathsf{T} + 4\mathsf{X} \qquad [\mathsf{G},\mathsf{M}]_1 = \partial\mathsf{X} \ .$$

The rest of the relevant operator product expansions are:

$$[\mathsf{P},\mathsf{K}]_2 = -3\mathsf{G} \qquad\qquad [\mathsf{P},\mathsf{K}]_1 = -3\mathsf{M} - \tfrac{3}{2}\partial\mathsf{G}$$
$$[\mathsf{P},\mathsf{M}]_2 = \tfrac{9}{2}\mathsf{K} \qquad\qquad [\mathsf{P},\mathsf{M}]_1 = 3(\mathsf{PG}) - \tfrac{1}{2}\partial\mathsf{K}$$
$$[\mathsf{X},\mathsf{K}]_2 = -3\mathsf{K} \qquad\qquad [\mathsf{X},\mathsf{K}]_1 = -3(\mathsf{PG})$$

and

$$[\mathsf{X},\mathsf{M}]_3 = -\tfrac{9}{2}\mathsf{G} \quad [\mathsf{X},\mathsf{M}]_2 = -5\mathsf{M} - \tfrac{9}{4}\partial\mathsf{G} \quad [\mathsf{X},\mathsf{M}]_1 = 4(\mathsf{XG}) + \tfrac{1}{2}\partial\mathsf{M} + \tfrac{1}{4}\partial^2\mathsf{G} \ .$$

We can obtain the remaining operator product expansions by using the associativity axiom. For example, to compute $[\mathsf{K},\mathsf{M}]_p$ we use the fact that $\mathsf{K} = [\mathsf{G},\mathsf{P}]_1$ and that $[\mathsf{G},-]_1$ is an odd derivation over all the brackets:

$$\begin{aligned}
[\mathsf{K},\mathsf{M}]_p &= [[\mathsf{G},\mathsf{P}]_1,\mathsf{M}]_p \\
&= [\mathsf{G},[\mathsf{P},\mathsf{M}]_p]_1 + [\mathsf{P},[\mathsf{G},\mathsf{M}]_1]_p \\
&= [\mathsf{G},[\mathsf{P},\mathsf{M}]_p]_1 + [\mathsf{P},\partial\mathsf{X}]_p \\
&= [\mathsf{G},[\mathsf{P},\mathsf{M}]_p]_1 + (p-1)[\mathsf{P},\mathsf{X}]_{p-1} + \partial[\mathsf{P},\mathsf{X}]_p \ ;
\end{aligned}$$

whence from the above formulae we find:

$$[\mathsf{K},\mathsf{M}]_3 = -15\mathsf{P} \quad [\mathsf{K},\mathsf{M}]_2 = -\tfrac{11}{2}\partial\mathsf{P} \quad [\mathsf{K},\mathsf{M}]_1 = 3(\mathsf{GK}) - 6(\mathsf{TP}) \ ,$$

which corrects a typo in equation (1.8) in the first appendix of [10]. (The notation is the same as in [10] except that here we call P what they call Φ, and aside from the above typo, we are in perfect agreement with their results.)

In [10], this algebra was obtained in a free field representation in terms of seven free bosons and seven free fermions. As a consequence, associativity of the operator product expansion is guaranteed. Abstractly, however, the Jacobi-like identities in the above algebra are only satisfied modulo the ideal generated by the weight $\tfrac{7}{2}$ null field N defined by

$$\mathsf{N} = 4(\mathsf{GX}) - 2(\mathsf{PK}) - 4\partial\mathsf{M} - \partial^2\mathsf{G} \ , \tag{3}$$

and vanishing identically in the free field realisation of [10].

2.3 The algebra associated with $Spin(7)$ holonomy

We finally look at the case of $Spin(7)$ holonomy. Let M be an irreducible 8-dimensional manifold with $Spin(7)$ holonomy. The tangent vectors are in the spinorial representation of $Spin(7)$, which in this context we call T. The only singlets in $\bigwedge^* T$ are a self-dual 4-form Θ and its square. In the large volume

limit, Θ gives rise to a weight 2 field:

$$X = \tfrac{1}{4!}\Theta_{ijkl}\psi^i\psi^j\psi^k\psi^l \ ,$$

which generates an extension of the $N{=}1$ Virasoro algebra.

Again let T and G denote the generators of the $N{=}1$ Virasoro algebra with central charge $c{=}12$. We now let X be a weight 2 field which we will choose *not* to be primary, and $\mathsf{M} = [\mathsf{G},\mathsf{X}]_1$ be its superpartner of weight $\tfrac{5}{2}$ but also not primary. The following operator product expansions define the algebra:

$$
\begin{array}{lll}
 & [\mathsf{G},\mathsf{X}]_2 = \tfrac{1}{2}\mathsf{G} & [\mathsf{G},\mathsf{X}]_1 = \mathsf{M} \\
[\mathsf{X},\mathsf{X}]_4 = 16 & [\mathsf{X},\mathsf{X}]_2 = 16\mathsf{X} & [\mathsf{X},\mathsf{X}]_1 = 8\partial\mathsf{X} \\
[\mathsf{G},\mathsf{M}]_4 = 4 & [\mathsf{G},\mathsf{M}]_2 = -\mathsf{T} + 4\mathsf{X} & [\mathsf{G},\mathsf{M}]_1 = \partial\mathsf{X} \\
[\mathsf{X},\mathsf{M}]_3 = -\tfrac{15}{2}\mathsf{G} & [\mathsf{X},\mathsf{M}]_2 = 8\mathsf{M} - \tfrac{15}{4}\partial\mathsf{G} & [\mathsf{X},\mathsf{M}]_1 = -6(\mathsf{GX}) + \tfrac{11}{2}\partial\mathsf{M} - \tfrac{5}{4}\partial^2\mathsf{G} \ .
\end{array}
$$

Again we can compute all other operator products from these by associativity. Notice that the field $\mathsf{T}' = \tfrac{1}{8}\mathsf{X}$ obeys a Virasoro algebra with central charge $c'{=}\tfrac{1}{2}$, corresponding to the Ising model. In counterpoint to the G_2 algebra, this one obeys associativity abstractly and not modulo an ideal. The reason is that the algebra admits a one-parameter (the central charge) deformation with the same fields. To see this we simply change basis to primary fields $\tilde{\mathsf{X}} = \mathsf{X} - \tfrac{1}{3}\mathsf{T}$ and $\tilde{\mathsf{M}} = \mathsf{M} - \tfrac{1}{6}\partial\mathsf{G}$. Then $\tilde{\mathsf{X}}$ is a superconformal primary of weight 2, and by the results of [4] there exists a unique such extension of the $N{=}1$ Virasoro algebra, which exists for generic values of the central charge. In fact, if we further rescale $\tilde{\mathsf{X}}$ to $\hat{\mathsf{X}} = \pm\tfrac{3}{\sqrt{23}}\tilde{\mathsf{X}}$, and define $\hat{\mathsf{M}} = [\mathsf{G},\hat{\mathsf{X}}]_1$ as its superpartner, then the new algebra satisfied by $\{\mathsf{T},\mathsf{G},\hat{\mathsf{X}},\hat{\mathsf{M}}\}$ agrees with the one in the appendix of the second reference in [4] for $c{=}12$.

3 New realisations

In this section we come to the main results of this note. We will construct new realisations of the G_2 and $Spin(7)$ superconformal algebras in sections 2.2 and 2.3 in terms of the algebras of sections 2.1.1 and 2.1.2, respectively.

Let K be a Calabi–Yau n-fold admitting an antiholomorphic involution σ. Then on the Kähler form ω and the (anti)holomorphic n-forms Ω and $\bar{\Omega}$, we have $\sigma^*\omega = -\omega$ and $\sigma^*\Omega = -\bar{\Omega}$. At the level of the algebra associated to such a manifold, the involution is represented by an automorphism which fixes T, G, A and C, and changes the sign of J, G', B and D. It is easy to check that this is an automorphism of the corresponding superconformal algebras. As a result, the subspace of the chiral algebra fixed by this automorphism will be a subalgebra.

In the case $n=4$ we see that this subalgebra contains the $Spin(7)$ superconformal algebra, generated by T and G together with

$$\mathsf{X} \equiv \mathsf{A} - \tfrac{1}{2}(\mathsf{JJ}) \qquad \text{and} \qquad \mathsf{M} \equiv \mathsf{C} - (\mathsf{JG'}) + \tfrac{1}{2}\partial\mathsf{G} \ .$$

All the brackets in section 2.3 are obeyed on the nose, except for $[\mathsf{X}, \mathsf{M}]_1$ (and hence $[\mathsf{M}, \mathsf{M}]_1$) which are obeyed only modulo the ideal generated by the null fields in (2). Indeed, if we define $\mathsf{R} \equiv [\mathsf{X}, \mathsf{M}]_1 + 6(\mathsf{GX}) - \tfrac{11}{2}\partial\mathsf{M} + \tfrac{5}{4}\partial^2\mathsf{G}$, we get that

$$
\begin{aligned}
\mathsf{R} &= 6(\mathsf{GA}) - 4(\mathsf{G'B}) + 2(\mathsf{JD}) - 2\partial\mathsf{C} \\
&= 4[\mathsf{G}, \mathsf{N}_{\mathrm{CY}}^{(1)}]_1 - 6[\mathsf{G'}, \mathsf{N}_{\mathrm{CY}}^{(2)}]_1 \ .
\end{aligned}
$$

In the case $n=3$ we need an auxiliary boson-fermion pair (j, ψ) normalised to $[j, j]_2 = 1$ and $[\psi, \psi]_1 = 1$ corresponding to the circle. This algebra has an automorphism given by $(j, \psi) \mapsto (-j, -\psi)$, which fixes the generators of an $N=1$ Virasoro algebra with central charge $\tfrac{3}{2}$:

$$
\begin{aligned}
\mathsf{T}_{S^1} &\equiv \tfrac{1}{2}(jj) + \tfrac{1}{2}(\partial\psi\psi) \\
\mathsf{G}_{S^1} &\equiv (j\psi) \ .
\end{aligned}
$$

Let K be a Calabi–Yau 3-fold admitting an antiholomorphic involution σ', and let σ be the involution on $K \times S^1$ acting by $\sigma(z, \theta) = (\sigma'(z), -\theta)$. On the superconformal algebra corresponding to $K \times S^1$, generated by T_{CY}, G_{CY}, J, $\mathsf{G'}$, A, B, C, D, j and ψ, the involution is represented by the automorphism which fixes T_{CY}, G_{CY}, A and C, and changes the signs of the other generators. If we define

$$\mathsf{T} \equiv \mathsf{T}_{\mathrm{CY}} + \mathsf{T}_{S^1} \qquad \mathsf{G} \equiv \mathsf{G}_{\mathrm{CY}} + \mathsf{G}_{S^1} \qquad \text{and} \qquad \mathsf{P} \equiv \mathsf{A} + (\mathsf{J}\psi) \ ,$$

then the other fields of the G_2 algebra follow:

$$
\begin{aligned}
\mathsf{X} &\equiv (\mathsf{B}\psi) + \tfrac{1}{2}(\mathsf{JJ}) - \tfrac{1}{2}(\partial\psi\psi) \\
\mathsf{K} &\equiv \mathsf{C} + (\mathsf{J}j) + (\mathsf{G'}\psi) \\
\mathsf{M} &\equiv (\mathsf{D}\psi) - (\mathsf{B}j) + (j\partial\psi) + (\mathsf{JG'}) - \tfrac{1}{2}\partial\mathsf{G} \ ;
\end{aligned}
$$

and computing their brackets, we find those of the G_2 superconformal algebra in section 2.2, modulo the ideal generated by the fields in (2). Also, as expected, the null field N in (3), belongs to the ideal generated by the fields in (2).

The above embeddings are not unique, of course, since we can still perform the automorphism mentioned at the end of section 2.1, namely $\mathsf{A}+i\mathsf{B} \mapsto e^{i\theta}(\mathsf{A}+i\mathsf{B})$ and similarly with $\mathsf{C} + i\mathsf{D}$.

9

Acknowledgements

References

[1] K Becker, M Becker, DR Morrison, H Ooguri, Y Oz and Z Yin, *Supersymmetric cycles in exceptional holonomy manifolds and Calabi-Yau 4-folds*, `hep-th/9608116`.

[2] MJ Duff, *M-Theory (the theory formerly known as strings)*, `hep-th/9608117`.

[3] R Blumenhagen, W Eholzer, A Honecker and R Hübel, *New N=1 extended superconformal algebras with two and three generators*, Int. J. Mod. Phys. **A7** (1992) 7841–7871.

[4] JM Figueroa-O'Farrill and S Schrans, *Extended superconformal algebras*, Phys. Lett. **257B** (1991) 69–73; and *The conformal bootstrap and super* W*-algebras*, Int. J. Mod. Phys. **A7** (1992) 591–628.

[5] PS Howe and G Papadopoulos, *Holonomy groups and* W*-symmetries*, `hep-th/9202036`, Comm. Math. Phys. **151** (1993) 467–480.

[6] DD Joyce, *Compact riemannian 7-manifolds with holonomy G_2: I*, J. Diff. Geometry **43** (1996) 291–328; and *II*, J. Diff. Geometry **43** (1996) 329–375.

[7] DD Joyce, *Compact 7-manifolds with holonomy Spin(7)*, Invent. math. **123** (1996) 507–552.

[8] S Mallwitz, *On SW minimal models and N=1 supersymmetric quantum Toda field theories*, `hep-th/9405025`, Int. J. Mod. Phys. **A10** (1995) 977–1003

[9] S Odake, *Extension of N=2 superconformal algebra and Calabi–Yau compactification*, Mod. Phys. Lett. **A4** (1989) 557-568.

[10] SL Shatashvili and C Vafa, *Superstrings and Manifolds of Exceptional Holonomy*, `hep-th/9407025`, Selecta Math. **A1** (1995) 347–381.

[11] K Thielemans, *An algorithmic approach to operator product expansions,* W*-algebras and* W*-strings*, `hep-th/9506159`.

[12] K Thielemans, *A Mathematica package for computing operator product expansions*, Int. J. Mod. Phys. **C2** (1991) 787-798;

[13] C Vafa, *Evidence for F-theory*, `hep-th/9602022`, Nucl. Phys. **B469** (1996) 403–418.

[14] C Vafa, *Modular invariance and discrete torsion on orbifolds*, Nucl. Phys. **B273** (1986) 592–606.

[15] MY Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989) 59–68.